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ON ESSENTIALLY NONLINEAR DYNAMICS OF ARCHES AND RINGS*

V.N. PILIPCHUK

An analytic method of investigating the essentially nonlinear dynamics of thin-walled, shallow arches and rings is given. The particular features of this method were illustrated earlier in /1/ on the example of a two-mass model of an arch. Below the basic transformations are generalized to embrace continuous systems.

A shallow arch represents apparently the simplest continuous object capable of snapping-through. In spite of this, no analytic methods exists of solving the corresponding Cauchy problem even in the case when only two degrees of freedom are taken into account. Below it is shown that the recognition of specific feature of the thin-walled constructions makes it possible to obtain sufficiently simple and obvious solutions suitable for large, as well as small amplitude motions. Such solutions describe, together with the oscillations about a single position of equilibrium, also the nonlocal processes of snapping-through of the arches. In addition they determine the corresponding conditions of dynamic stability "in the large".

1. We write the basic equation of motion of a shallow arch (Fig.1) in the form /2/

$$A_s\rho\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial y^4}-T\frac{\partial^2 w}{\partial y^2}=q(t,y) \tag{1.1}$$

$$T=\frac{EA_s}{2l}\int_0^l\left[\left(\frac{\partial w}{\partial y}\right)^2-\left(\frac{\partial w_0}{\partial y}\right)^2\right]dy$$

Here $w=w(t,y)$ and $w_0=w_0(y)$ denote the coordinates of the deformed and initial central line of the arch respectively, ρ is the mass density, $q(t,y)$ is a function of transverse load, T is the magnitude of the thrust, EI is the flexural rigidity and EA_s is the tension-compression rigidity. Let us introduce the parameter a_1 characterizing the arch height, and the following dimensionless quantities for future convenience:

$$a_1=\frac{2}{l}\int_0^l w_0\sin\frac{\pi}{l}y\,dy,\quad \tau=\left(\frac{\pi}{l}\right)^2\sqrt{\frac{E}{\rho}}\,a_1t,\quad \eta=\frac{\pi}{l}y,\quad \lambda=\frac{EI}{a_1^2EA_s}$$

$$W=W(\tau,\eta)=\frac{w}{a_1},\quad W_0=W_0(\eta)=\frac{w_0}{a_1},\quad P=P(\tau,\eta)=\left(\frac{\pi}{l}\right)^4\frac{q}{a_1EI}$$

The equation (1.1) now becomes

$$\frac{\partial^2 W}{\partial\tau^2}+\lambda\frac{\partial^4(W-W_0)}{\partial\eta^4}-f[W]\frac{\partial^2 W}{\partial\eta^2}=\lambda P \tag{1.2}$$

$$f[W]=\frac{1}{2\pi}\int_0^\pi\left[\left(\frac{\partial W}{\partial\eta}\right)^2-\left(\frac{\partial W_0}{\partial\eta}\right)^2\right]d\eta$$

where the functional $f[W]$ determines the magnitude of the thrust referred to $EA_s(a_1\pi/l)^2$. We shall consider, for definiteness, a doubly hinged arch, and write the system of the boundary and initial conditions in the form

$$\left.\frac{\partial^2(W-W_0)}{\partial\eta^2}\right|_{\eta=0}=\left.\frac{\partial^2(W-W_0)}{\partial\eta^2}\right|_{\eta=\pi}=0,\quad W(\tau,0)=W(\tau,\pi)=0 \tag{1.3}$$

$$\left.\frac{\partial W}{\partial\tau}\right|_{\tau=0}=W^{\cdot}(0,\eta),\quad W|_{\tau=0}=W(0,\eta)$$

We shall restrict our investigations to the thin-walled systems, for which the following relation holds (h is the arch thickness):

$$\lambda=\frac{1}{a_1^2}\frac{EI}{EA_s}\sim\left(\frac{h}{a_1}\right)^2\ll 1$$

Direct application of the methods of perturbing the parameter λ to solution of the problem

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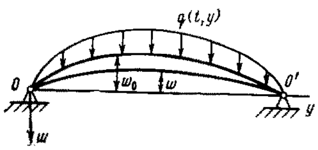

Fig.1

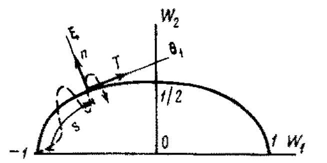

Fig.2

(1.2), (1.3) is not easy, since the limiting ($\lambda = 0$) equation (1.2) remains essentially nonlinear. (Additional constraints are not imposed on the magnitude of the displacements in order to be able to analyze the global dynamic modes with snap-through). Because of this, we shall first transform (1.2).

We describe the motion of the arch in terms of a trajectory of the representative point in the configurational space

$$X_\infty = \Big\{W(\eta) \mid W(\eta) \in C^4[0,\pi], \quad W(0) = W(\pi) = 0$$
$$\frac{d^2(W - W_0)}{d\eta^2}\Big|_{\eta=0} = \frac{d^2(W - W_0)}{d\eta^2}\Big|_{\eta=\pi} = 0\Big\}$$

in which the scalar product and Euclidean norm are defined as follows:

$$\langle XY\rangle_\eta = \frac{1}{\pi}\int_0^\pi XY\,d\eta; \quad X, Y \in X_\infty; \quad |X| = \sqrt{\langle XX\rangle_\eta}$$

The relation

$$f[W] = 0 \tag{1.4}$$

defines a certain manifold in the space X_∞. Thus, taking into account two degrees of freedom (two forms of linear oscillations, see Sect.2) we find that (1.4) assumes a form of an ellipse equation (2.2) in the configurational plane, and for N degrees of freedom, a hyper-ellipsoid of dimension $N - 1$. When $\lambda = 0$, (1.2) implies that the manifold (1.4) represents a geometrical site of the equilibrium configurations of a perfectly flexible arch (string).

Let us parametrize (1.4) by introducing a certain curvilinear orthogonal coordinate system $S = \| s_1, s_2, \ldots, s_i \ldots \|$ (s_i is the arc length of the i-th coordinate line), which will be later defined more sharply, and denote the point lying on the manifold (1.4) by $W^\circ = W^\circ(S, \eta)$. Thus we have $f[W^\circ] \equiv 0$. We introduce in the space X_∞ a local coordinate system $H = \| \xi, \Theta \|^T$ with the origin at the point W°. We direct the coordinate ξ along the normal to (1.4), and place the coordinates $\Theta = \| \theta_1 \theta_2 \ldots \|$ in the tangent hyperplane Fig.2). The system of the corresponding orthonormal basic vectors has the form (n denotes the unit vector of the normal to (1.4), and T are tangent vectors)

$$L = L(S) = \| n, T \|, n = -\frac{\partial^2 W^\circ}{\partial\eta^2}\left|\frac{\partial^2 W^\circ}{\partial\eta^2}\right|^{-1},$$
$$T = \frac{\partial W^\circ}{\partial S} = \left\| \frac{\partial W^\circ}{\partial s_1}, \frac{\partial W^\circ}{\partial s_2}, \ldots \right\|$$

The orthogonality of n and T follows from the relation

$$\langle nT\rangle_\eta = \frac{1}{4\,|\partial^2 W^\circ/\partial\eta^2|}\frac{\partial f[W^\circ]}{\partial S} = 0$$

obtained by inegrating (1.3) by parts, which the boundary conditions taken into account.

We pass to the local coordinate system H using the relation

$$W(\eta) = W^\circ(S, \eta) + \lambda L(S) H = W^\circ(S, \eta) + \lambda\left(n\zeta + \frac{\partial W^\circ}{\partial s_1}\theta_1 + \frac{\partial W^\circ}{\partial s_2}\theta_2 + \ldots\right)$$

and study thus a certain λ-neighborhood of the manifold (1.4). In accordance with the statement that the deformation of a thin-walled construction under large displacement is basically a flexural deformation, we assume that the trajectories of the system in motion are situated in this neighborhood. Such an assumption is based on the energetic estimates /1/ and on the fact that the motion in the direction of the normal n is related to the tension-compression type deformation, and the motion along the manifold (1.4) is associated with the flexure of the central arch line.

Let the local coordinate system H follow in the course of motion the representative point, and let $S = S(t^\circ), t^\circ = \sqrt{\lambda}\tau$. We write the equation of the point trajectory in the form

$$W(\tau, \eta) = W^\circ(S, \eta) + \lambda L(S) H(\tau) \tag{1.5}$$

Substituting (1.5) into (1.2) we obtain the following equation for H:

$$L\frac{d^2H}{d\tau^2}+n\omega_0{}^2\xi=F_0(W^\circ)-\lambda[F_1(H)+F_2(H)]-$$
$$\lambda^2\left[\frac{\partial^2L}{\partial t^{\circ 2}}H+F_3(H)\right];\quad \omega_0=\left|\frac{\partial^2W^\circ}{\partial\eta^2}\right| \tag{1.6}$$
$$F_0(W^\circ)=\frac{\partial^2W^\circ}{\partial t^{\circ 2}}+\frac{\partial^4(W^\circ-W_0)}{\partial\eta^4}-P,$$
$$F_1(H)=2\frac{\partial L}{\partial t^\circ}\frac{dH}{d\tau}+\frac{\partial^4L}{\partial\eta^4}H$$
$$F_2(H)=\frac{\omega_0}{2}\left\langle\left(\frac{\partial L}{\partial\eta}H\right)^2\right\rangle_\eta n-\omega_0\xi\frac{\partial^2L}{\partial\eta^2}H,$$
$$F_3(H)=\frac{1}{2}\left\langle\left(\frac{\partial L}{\partial\eta}H\right)^2\right\rangle_\eta\frac{\partial^2L}{\partial\eta^2}H$$

where we used the relation

$$f[W^\circ+\lambda LH]=\lambda\omega_0\xi+\frac{\lambda^2}{2}\left\langle\left(\frac{\partial L}{\partial\eta}H\right)^2\right\rangle_\eta$$

obtained with the boundary conditions taken into account.

Projecting the equation (1.6) onto the vector of the normal n and tangent hyperplane T, we obtain

$$\frac{d^2\xi}{d\tau^2}+\omega_0{}^2\xi=-\langle nF_0(W^\circ)\rangle_\eta-\lambda[\langle nF_1(H)\rangle_\eta+\langle nF_2(H)\rangle_\eta]-\lambda^2\left[\left\langle n\frac{\partial^2L}{\partial t^{\circ 2}}\right\rangle_\eta H+\langle nF_3(H)\rangle_\eta\right] \tag{1.7}$$
$$\frac{d^2\Theta}{d\tau^2}=-\langle TF_0(W^\circ)\rangle_\eta-\lambda[\langle TF_1(H)\rangle_\eta+\langle TF_2(H)\rangle_\eta]-\lambda^2\left[\left\langle T\frac{\partial^2L}{\partial t^{\circ 2}}\right\rangle_\eta H+\langle TF_3(H)\rangle_\eta\right]$$

The specific feature of the sysyem (1.7) quasilinear in ξ, Θ is the fact that a linearized system has only one principal frequency ω_0 different from zero, and the perturbations contain arbitrary functions $\|s_1(t^\circ), s_2(t^\circ),\ldots\|$ of "slow time". The system (1.7) can be solved in the framework of the quasilinear theory /3/ and functions $\|s_1(t^\circ), s_2(t^\circ),\ldots\|$ must be chosen so that the quantity $\Theta=\|\theta_1,\theta_2,\ldots\|$ remains bounded over the time τ. Satisfying the latter requirement, we put the right-hand part of the second equation of (1.7) averaged over τ, equal to zero

$$\langle TF_0(W^\circ)\rangle_\eta+\lambda[\langle T\langle F_1(H)\rangle_\tau\rangle_\eta+\langle T\langle F_2(H)\rangle_\tau\rangle_\eta]+$$
$$\lambda^2\left[\left\langle T\frac{\partial^2L}{\partial t^{\circ 2}}\right\rangle_\eta\langle H\rangle_\tau+\langle T\langle F_3(H)\rangle_\tau\rangle_\eta\right]=0 \tag{1.8}$$
$$\left(\langle\cdot\rangle_\tau=\lim_{l\to\infty}\frac{1}{l}\int_0^l(\cdot)\,d\tau\right)$$

This yields, in addition to the quasilinear equation (1.7), an essentially nonlinear system of equations (1.8) for the variables $S(t^\circ)=\|s_1(t^\circ), s_2(t^\circ),\ldots\|$ representing the coordinates of the point W° on the manifold $f[W]=0$. The corresponding initial conditions are given by the relations (1.3) and (1.5). To solve the equations (1.7), (1.8) in the configurational space X_∞ we must select a system of basis functions, and a system of normal modes of linear oscillations of the arch can serve as such.

Discretization of the system by means of the modal analysis can also be carried out directly in the initial equation (1.2). The relation (1.8) however yields, for a small number of the modes, a system of equations which is much simpler than the initial equation. This is due to the fact that firstly is has a lower dimension equal to that of the manifold (1.4) and, secondly, the processes described by it take place in the slow time t°. The additional high frequency components of the motion are described by the quasilinear equations (1.7). Such a separation of the motions facilitates the solution of the problem also when numerical methods are used for the case of a large number of essential modes.

2. We illustrate the approach described above on the example of a sinusoidal arch ($W_0(\eta)=-\sin\eta$) taking into account the first two modes of oscillation. The radius vector of the point lying on the manifold (1.4) is described, in this case, by the expression

$$W^\circ(S,\eta)=W_1{}^\circ(s)\sin\eta+W_2{}^\circ(s)\sin 2\eta \tag{2.1}$$

Substituting (2.1) into (1.4) we obtain an equation of an ellipse with semiaxes $a=1$, $b={}^1/_2$ on the configurational plane $W_1=2\langle\sin\eta\rangle_\eta$, $W_2=2\langle W\sin 2\eta\rangle_\eta$ (Fig.2). The coordinate system S in this case has a single component $s_1=s$, and we choose as this component the arc length of the ellipse counted from the top $W_1=-1$, $W_2=0$. We write for convenience the equation of the ellipse in the parametric form

$$W_1{}^\circ=-\cos\varphi,\quad W_2{}^\circ=\frac{1}{2}\sin\varphi;\quad ds=\frac{\omega_0}{\sqrt{2}}d\varphi;\quad \omega_0=\left[\frac{1}{2}(1+3\sin^2\varphi)\right]^{1/2} \tag{2.2}$$

The value $\varphi = 0$ corresponds to the initial undeformed position of the arch, and $\varphi = \pi$ to the overturned arch. We have

$$n = \frac{1}{\sqrt{2}\,\omega_0}(-\cos\varphi\sin\eta + 2\sin\varphi\sin 2\eta) \tag{2.3}$$
$$T = \frac{\partial W^\circ}{\partial s} = \frac{1}{\sqrt{2}\,\omega_0}(2\sin\varphi\sin\eta + \cos\varphi\sin 2\eta);\quad L = \| n, T \|$$

Let us apply to the arch at rest, at the instant $\tau = 0$ a constantly acting load $P = p\sin\eta$. Having solved the quasilinear system (1.7) we use, as in /1/, the method of two scale expansions to obtain in the basic approximation

$$\xi = r(\varphi) - r(0)\sqrt{\frac{\omega_0(0)}{\omega_0(\varphi)}}\cos t^*,\quad \theta_1 = 0 \tag{2.4}$$
$$r(\varphi) = \frac{1}{\sqrt{2}\,\omega_0^3}\left[\cos\varphi(1-p) - 1 - 15\sin^2\varphi + \left(\frac{d\varphi}{dt^\circ}\right)^2\right]$$
$$\frac{dt^*}{d\tau} = \omega_0(\varphi)$$

Relation (1.8) yields the following expression for the function $\varphi(t^\circ)$:

$$\frac{\omega_0}{2}\frac{d}{dt^\circ}\left(\omega_0\frac{d\varphi}{dt^\circ}\right) + f(\varphi, p) = 0;\quad f(\varphi, p) = (3\cos\varphi + 1 - p)\sin\varphi \tag{2.5}$$

Analyzing now the potential function

$$\Phi(\varphi, p) = \int_0^\varphi f(x, p)\,dx = \frac{3}{2}\sin^2\varphi + (1-p)(1-\cos\varphi)$$

corresponding to (2.5), we find on the segment $0 \leqslant \varphi \leqslant \pi$ the value $p = 4$ for which the function has an absolute maximum at the point $\varphi = 0$ and the system is unstable "in the large". The influence of the dynamic effects on the magnitude of the critical load becomes apparent in the next approximation. The corresponding correction to the function $f(\varphi, p)$ has the form

$$\frac{3\lambda}{2}\sin\varphi\cos\varphi\left[r^2(\varphi) + \frac{5\sqrt{2}}{\omega_0}r(\varphi) + \frac{r^2(0)}{2}\frac{\omega_0(0)}{\omega_0(\varphi)}\right]$$

and the value of the critical load is equal to $p = 4 + 24\lambda$ (in the case of quasistatic load we would have $p = 4 - 24\lambda$). The function $\varphi(t^\circ)$ describes, depending on the quantity p and initial conditions, the oscillations, either about a single position of equilibrium, or about several (snap-through) positions, and is obtained from (2.5) in terms of the quadratures

$$\frac{1}{2}\int_0^\varphi \frac{\omega_0(x)\,dx}{[\varphi^{\cdot 2}/8 - \Phi(x, p)]^{1/2}} = t^\circ;\quad \varphi^\cdot = \frac{d\varphi}{dt^\circ}\Big|_{t^\circ=0} = 2\frac{dW_2}{dt^\circ}\Big|_{t^\circ=0} \tag{2.6}$$

where φ is the velocity perturbation of the skew symmetric mode. The relation (2.6) together with (1.5), (2.1), (2.3) and (2.4) determine the solution sought.

3. Let us consider a problem of free oscillation of a thin-walled circular ring when the amplitudes are large. The correspondong equations describing the oscillations can be obtained within the framework of the theory of shallow shells /4/ from (1.2) by the change of variables $W = W_0 - w$:

$$\frac{\partial^2 w}{\partial\tau^2} + \lambda\frac{\partial^4 w}{\partial\eta^4} + f[w]\left(\frac{\partial^2 w}{\partial\eta^2} - 1\right) = 0$$
$$\lambda = \frac{D}{Eh} \sim h^2,\quad \tau = \sqrt{\frac{E}{\rho}}\,t,\quad f[w] = \left\langle w + \frac{1}{2}\left(\frac{\partial w}{\partial\eta}\right)^2\right\rangle_\eta;$$
$$w(\eta + 2\pi, \tau) = w(\eta, \tau)$$

Here the ring radius is assumed to be unity $(d^2W_0/d\eta^2 = 1)$, the quantity w is counted from the undeformed position of the central line in the direction of the outer normal, and the averaging is carried out over the whole length of the ring $0 \leqslant \eta \leqslant 2\pi$. Taking into account two oscillation modes, we put

$$w^\circ = W_0^\circ(s) + \sqrt{2}W_k^\circ(s)\cos k\eta,\ f[w^\circ] \equiv 0$$

The latter relation defines a parabola on the configurational plane

$$W_0^\circ = -\frac{1}{2k^2}\operatorname{tg}^2\varphi,\quad W_k^\circ = \frac{1}{k^2}\operatorname{tg}\varphi,\quad ds = \frac{d\varphi}{k^2\cos^3\varphi}$$

and in this case we must put in (1.7), (1.8)

$$\omega_0 = |1 - \partial^2 w^\circ/\partial\eta^2|, \ n = \omega_0^{-1}(1 - \partial^2 w^\circ/\partial\eta^2)$$

Following Sect.2 we now obtain the solution and write it in its final form

$$w = -\frac{1}{2k^2}\operatorname{tg}^2\varphi + \frac{\sqrt{2}}{k^2}\operatorname{tg}\varphi\cos k\eta + \lambda\xi(\cos\varphi + \sqrt{2}\sin\varphi\cos k\eta) \tag{3.1}$$

$$\xi = \alpha_0(\varphi_0)\sqrt{\frac{\cos\varphi}{\cos\varphi_0}}\cos t^* + \frac{1}{k^2\cos\varphi}\left(\frac{d\varphi}{dt^\circ}\right)^2 - k^2\cos\varphi\sin^2\varphi$$

$$\alpha_0(\varphi_0) = \text{const}, \quad dt^*/d\tau = \omega_0 = 1/\cos\varphi$$

$$\left(t^\circ = -\int_{\varphi_0}^{\varphi}\frac{\cos^{-3}x\,dx}{\sqrt{2[\varnothing(\varphi_0) - \varnothing(x)]}}, \quad \varphi_0 = \varphi|_{t^\circ=0},\right.$$

$$\left.-\varphi_0 \leqslant \varphi \leqslant \varphi_0, \ \varnothing(\varphi) = \frac{1}{8}\operatorname{tg}^4\varphi + \frac{k^4}{2}\operatorname{tg}^2\varphi\right)$$

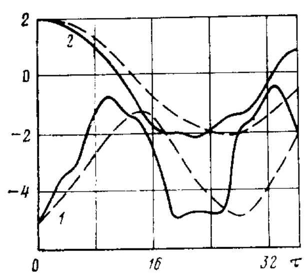

Fig.3

where the function $\varphi(t^\circ)$ is expressed by a quadrature given in brackets.

Fig.3 depicts the solution (3.1) in solid lines, and the quasilinear solution /5/ in dashed lines. On the ordinate axis we have $W_0 = 10^3\langle w\rangle_\eta$ (curves *1*) and $W_1 = 10^2\sqrt{2}\langle w\cos 4\eta\rangle_\eta$ (curves *2*). A solution obtained numerically using the Kutta–Runge method with the integration step $\Delta\tau = 0,1$ coincides for $\lambda = 10^{-3}$, $k = 4$ with (3.1) in Fig.3. Analysis of the results depicted in Fig.3 shows the effectiveness of the method proposed.

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